MATH20132 Calculus of Several Variables. 2019-20

Problems 6: Graphs, level sets, parametric sets, Implicit & Inverse functions

Surfaces as a level set.

1. Are the following level sets surfaces? (Look at the Jacobian matrices of the level sets).

i. $\{\mathbf{x} \in \mathbb{R}^3 : x^2 + 3y^2 + 2z^2 = 9\}$,

ii. The set of $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$\begin{aligned} x^2 + y^2 - z^2 &= 1, \\ x^2 + 3y^2 + 2z^2 &= 9. \end{aligned}$$

iii. The set of $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$\begin{aligned} x^2 + y^2 - z^2 &= 11, \\ x^2 + 3y^2 + 2z^2 &= 9. \end{aligned}$$

iv. The set of $\mathbf{x} \in \mathbb{R}^4$ satisfying

$$\begin{aligned} 3x + 2y^2 + u^2 + v^2 &= 13, \\ x^3 - y^3 + u^3 - v^3 &= 0, \\ 3x^3 + 5y + 5u^2 - v^2 &= 24. \end{aligned}$$

Hint the point $\mathbf{p} = (1, 1, 2, 2)^T$ may be of interest.

Surfaces as an image set.

2. Are the following parametrically defined sets surfaces? Give your reasons. (Look at their Jacobian matrices.)

i. $\left\{ (x^2 + y^2,\, xy,\, 2x - 3y)^T : x, y \in \mathbb{R} \right\}$,

ii. $\left\{ (x^2 + y^2,\; xy,\, 2x^3 - 3y^2)^T : (x, y) \in \mathbb{R}^2 \setminus \{\mathbf{0}\} \right\}$,

iii. $\left\{ (x^2 + y^2,\, xy,\, 2x^3 - 3y^2)^T : x > 0, y > 0 \right\}$,

iv. $\left\{ (ye^x,\ xe^y,\ 1)^T : x, y \in \mathbb{R} \right\}$.

Graphs in $\mathbb{R}^3$.

3. Suppose that $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is Fréchet differentiable on U. Let

$$G_f = \left\{ \begin{pmatrix} \mathbf{a} \\ f(\mathbf{a}) \end{pmatrix} : \mathbf{a} \in U \right\} \subseteq \mathbb{R}^3.$$

be the graph of f.

Prove that as $\mathbf{a} \in U$ varies in the $\mathbf{v} \in \mathbb{R}^2$ direction the directional derivative $d_{\mathbf{v}} f(\mathbf{a})$ represents the rate of change in the z-coordinate of the corresponding points on the graph.

Hint look at the rate of change of going from

$$\begin{pmatrix} \mathbf{a} \\ f(\mathbf{a}) \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \mathbf{a} + t\mathbf{v} \\ f(\mathbf{a} + t\mathbf{v}) \end{pmatrix}$$

4. Let $f(\mathbf{x}) = 4 - 3x^2 + xy - y^2, \mathbf{x} \in \mathbb{R}^2$. If a spider stands on the graph of f above $\mathbf{q} = (1, 1)^T$ in which direction should the spider move for

i. the fastest ascent?

ii. the fastest descent?

iii. to stay at the same height?

Remember, though the graph lies within $\mathbb{R}^3$ the direction will be in $\mathbb{R}^2$; we see this in real life when, on a mountain, you only give directions using West & North coordinates, no mention is given of up or down.

Hint Look back at Question 9 on Sheet 5 that looked at bounds on $d_{\mathbf{v}} f(\mathbf{a})$ and when they are attained.

5. Define the function

$$f(\mathbf{x}) = (x - 1)^2 + y^2 \quad \text{for} \quad \mathbf{x} = (x, y)^T \in \mathbb{R}^2.$$

Imagine standing on the graph of f above the point $\mathbf{q} = (0, 2)^T$ and spilling water. In which direction would the water flow?

Graphs as an image set and a level set

6. Define $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^2, (x, y)^T \to (xy^2, x^2 + y)^T$.

i. The graph G_ϕ is the image of some function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^4$. Find $\mathbf{F}$ and the Jacobian matrix $J\mathbf{F}(\mathbf{x})$.

ii. The graph G_ϕ can be expressed as a level set of a system of equations. Find such a system of equations and find the Jacobian matrix of the system.

Hint Since $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^4$ write $\mathbf{F}(\mathbf{x}) = (s, t, u, v)^T$ and find relations between the s, t, u and v.

Linear Algebra

Vector subspaces in $\mathbb{R}^n$.

7. In the notes it is stated that

i. if $M \in M_{n,r}(\mathbb{R})$ then $\{M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\}$ is a vector subspace of $\mathbb{R}^n$;

ii. if $N \in M_{m,n}(\mathbb{R})$ then $\{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\}$ is a vector subspace of $\mathbb{R}^n$;

iii. if $S \subseteq \mathbb{R}^n$ then the orthogonal complement

$$S^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \bullet \mathbf{s} = 0 \text{ for all } \mathbf{s} \in S\}$$

is a vector subspace of $\mathbb{R}^n$.

Prove all these assertions.

Planes in $\mathbb{R}^n$.

8. i. A plane in $\mathbb{R}^3$ is given parametrically by

$$\left\{ \begin{pmatrix} 2x + 4y - 5 \\ 2x + y - 2 \\ 2x - 3y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

Express this plane as

a. a graph

$$\left\{ \begin{pmatrix} \mathbf{u} \\ \phi(\mathbf{u}) \end{pmatrix} : \mathbf{u} \in \mathbb{R}^2 \right\},$$

of some function $\phi : \mathbb{R}^2 \to \mathbb{R}$,

b. a level set,

$$f^{-1}(0) = \left\{ \mathbf{s} \in \mathbb{R}^3 : f(\mathbf{s}) = 0 \right\}.$$

for some $f : \mathbb{R}^3 \to \mathbb{R}$.

ii. Repeat for the parametric set

$$\left\{ \begin{pmatrix} 2x + 2y - 2 \\ x + y - 1 \\ 2x - 3y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

iii. Repeat for

$$\left\{ \begin{pmatrix} 4x - 4y + 8 \\ -2x + y - 1 \\ 3x - 4y + 6 \\ 4y - 4 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\},$$

this time expressing this as a graph of some function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, and then as a level set.

Level sets are locally graphs

9 i. State the Implicit Function Theorem.

ii. a. Prove, using the Implicit Function Theorem, that for the solutions $(x, y, u, v)^T \in \mathbb{R}^4$ of

$$\begin{aligned} x^2 + y^2 + 2uv &= 4 \\ x^3 + y^3 + u^3 - v^3 &= 0, \end{aligned}$$

there exists an open subset of $\mathbb{R}^4$ containing the solution $\mathbf{p} = (-1, 1, 1, 1)$ in which the u and v can be given as functions of x and y, with $(x, y)^T$ in some open subset of $\mathbb{R}^2$ containing the point $\mathbf{q} = (-1, 1)^T$.

b. Find the partial derivatives of u and v with respect to x and y at $\mathbf{q}$.

c. Is there any open subset of $\mathbb{R}^4$ containing $\mathbf{p}$ in which y and u can be given as functions of x and v? What happens if you attempt to find the partial derivatives of y and u as functions of x and v at this point?

iii. Do the same calculation of partial derivatives for the point $(-1, 1, -1, -1)^T$.

10. Show that the following level sets are locally graphs around the point given.

i. $(x, y, z)^T \in \mathbb{R}^3 : xy^2z^3 - x^2y^2z^2 + x^3y^2 = 18$ with $\mathbf{p} = (2, 3, -1)^T$.

ii. $(x, y, z)^T \in \mathbb{R}^3$:

$$\begin{aligned} x^2 + 3y^2 + 2z^2 &= 9, \\ xyz &= -2, \end{aligned}$$

with $\mathbf{p} = (2, -1, 1)^T$.

11i. Does the equation $x = \sin(xyz)$ determine x as a function of y and z in any open subset of $\mathbb{R}^3$ containing the point $\mathbf{p} = (1, 1, \pi/2)^T$, i.e. as a graph $x = \phi(y, z)$?

ii. Does the equation $x = \sin(xyz)$ determine z as a function of x and y in a open subset of $\mathbb{R}^3$ containing the point $\mathbf{p} = (1, 1, \pi/2)^T$, i.e. as a graph $z = \phi(x, y)$?

Additional Questions 6

12. Prove part of a Theorem from the Notes: $P \subseteq \mathbb{R}^n$ is a plane of dimension r iff

i. there exists a point $\mathbf{p} \in \mathbb{R}^n$ and a full rank matrix $M \in M_{n,r}(\mathbb{R})$ such that $P = \{\mathbf{p} + M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\}$,

ii. there exists a point $\mathbf{p} \in \mathbb{R}^n$ and a full rank matrix $N \in M_{n-r,n}(\mathbb{R})$ such that $P = \{\mathbf{x} \in \mathbb{R}^n : N(\mathbf{x} - \mathbf{p}) = \mathbf{0}\}$.

Hint for part ii. If $\mathcal{V} \subseteq \mathbb{R}^n$ is a vector space then $\dim \mathcal{V}^\perp = n - \dim \mathcal{V}$. (For a proof see appendix of Section 3 Part 1.)

The important part of these results is the relationship between the dimension of the plane and the fact that the matrices are of full rank

Solution i. By Question 7 $\{M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\}$ is a vector space and as stated in the notes

$$\{M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\} = \operatorname{span}\{\mathbf{c}_1, ..., \mathbf{c}_r\} \tag{1}$$

where the $\mathbf{c}_i$ are the columns of M. Then

$$\begin{aligned} M \text{ is of full rank} \quad &\text{iff} \quad \dim \operatorname{span}\{\mathbf{c}_1, ..., \mathbf{c}_r\} = r \\ &\text{iff} \quad \dim\{M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\} = r \quad \text{by (1)} \\ &\text{iff} \quad \{\mathbf{p} + M\mathbf{t} : \mathbf{t} \in \mathbb{R}^r\} \quad \text{is a plane of dimension } r. \end{aligned}$$

ii. By Question 7 $\{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\}$, is a vector space and as stated in the notes

$$\{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\} = \operatorname{span}\{\mathbf{r}_1, ..., \mathbf{r}_{n-r}\}^\perp \tag{2}$$

where the $\mathbf{r}_i$ are the rows of N. Then

$$\begin{aligned} N \text{ is of full rank} \quad &\text{iff} \quad \dim \operatorname{span}\{\mathbf{r}_1, ..., \mathbf{r}_{n-r}\} = n - r \\ &\text{iff} \quad \dim \operatorname{span}\{\mathbf{r}_1, ..., \mathbf{r}_{n-r}\}^\perp = n - (n - r) \quad \text{by hint} \\ &\text{iff} \quad \dim\{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \mathbf{0}\} = r \quad \text{by (2)} \\ &\text{iff} \quad \{\mathbf{x} \in \mathbb{R}^n : N(\mathbf{x} - \mathbf{p}) = \mathbf{0}\} = \mathbf{p} + \{\mathbf{y} \in \mathbb{R}^n : N\mathbf{y} = \mathbf{0}\} \end{aligned}$$

is a plane of dimension r.

13 Let $\phi(\mathbf{u}) = 2u^2 + 3uv - 4v^2$ for $\mathbf{u} = (u, v)^T \in \mathbb{R}^2$. Then $\mathbf{p} = (1, 2, -8)^T$ is a point on the graph of ϕ. In which direction $\mathbf{v} \in \mathbb{R}^2$ is the fastest ascent? the fastest descent? no change in height?

14. Define the function

$$f(\mathbf{x}) = \frac{x^2y + 2xy^2}{1 + x^2 + y^2} \quad \text{for} \quad \mathbf{x} = (x, y)^T \in \mathbb{R}^2.$$

Imagine standing on the graph of f above the point $\mathbf{q} = (1, 2)^T$ and spilling water. In which direction would the water flow?

15. Let $T : \mathbb{R}^3 \to \mathbb{R}, x \mapsto 10 - 2e^{2x^2+3y^2+z^2}$ give the temperature at each point in $\mathbb{R}^3$.

i. In which direction from $\mathbf{p} = (2, 0, 2)^T$ does the temperature increases as quickly as possible? Decreases as quickly as possible?

ii. Let $S \subseteq \mathbb{R}^3$ be a surface in $\mathbb{R}^3$ given parametrically as

$$\left\{ \begin{pmatrix} u^2 + v \\ u - v \\ uv + u \end{pmatrix} : 0 \leq u, v \leq 2 \right\}.$$

The point $\mathbf{p} = (2, 0, 2)^T \in S$ is the image of $\mathbf{q} = (1, 1)^T$. If a spider stands at $\mathbf{p}$, and is restricted to stay **on** the surface, in which direction must they move to increase the temperature as quickly as possible; to decrease it as quickly as possible?

16. i. Prove that

$$\begin{aligned} xe^y + uz - \cos(v\pi/2) &= 2 \\ u\cos(y\pi/2) + x^2v - yz^2 &= 1, \end{aligned}$$

can be solved for u, v in terms of x, y, z near $\mathbf{p} = (2, 0, 1, 1, 0)^T$. (The general point of $\mathbb{R}^5$ is $(x, y, z, u, v)^T$).

ii. Can you find a point $\mathbf{p}'$ around which the system can be solved for x and z in terms of y, u and v?

17. Can $(x^2 + y^2 + 2z^2)^{1/2} = \cos z$ be solved for y in terms of x and z near $(0, 1, 0)^T$?